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Lifshitz Tails for Periodic Plus Random Potentials

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We prove that the integrated density of states $\rho(\lambda)$ for a potential $W_\omega = V_{\text{per}} + V_\omega$ has Lifshitz tails where V_{per} is a periodic potential with reflection symmetry and V_ω is a random potential, e.g., of the form $V_\omega = \sum q_i(\omega) f(x-i)$.

KEY WORDS: Lifshitz tails; Anderson model; Dirichlet–Neumann bracketing.

1. INTRODUCTION

We consider a random Schrödinger operator

$$H_\omega = H_0 + V_{\text{per}} + V_\omega$$

on $L^2(\mathbb{R}^\nu)$, where $H_0 = -\Delta$, V_{per} is a periodic potential, and V_ω is a random potential of the form

$$V_\omega(x) = \sum_{i \in \mathbb{Z}^\nu} q_i(\omega) f(x-i) \tag{1}$$

f is a nonnegative function with $f(x) = O(|x|^{-\nu-\varepsilon})$ as $|x| \to \infty$, $f \in L^p(\mathbb{R}^\nu)$ with $p = 2$ for $\nu \leqslant 3$, $p > 2$ for $\nu = 4$, $p = \nu/2$ for $\nu \geqslant 5$. $\{q_i\}_{i \in \mathbb{Z}^\nu}$ are independent random variables with a common distribution P_0. We assume that the support $\operatorname{supp} P_0$ of P_0 is compact, is not a single point, and $0 = \inf \operatorname{supp} P_0$. Moreover, $P_0([0, \varepsilon)) \geqslant C\varepsilon^N$ for some $C, N > 0$.

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$V_{\rm per}$ is a $\mathbb{Z}^\nu$-periodic function that is locally L^p (p as above), and $V_{\rm per}$ is invariant under reflections of the coordinate axes, i.e., with $R_i(x_1,..., x_\nu) = (x_1,..., x_{i-1}, -x_i, x_{i+1},..., x_\nu)$ we have $V(R_i x) = V(x)$ for $i = 1,..., \nu$. For example, if $V_{\rm per}(x) = \sum f_0(x - i)$, then $V_{\rm per}$ is reflection invariant if f_0 is spherically symmetric (or reflection invariant itself).

Instead of taking $\mathbb{Z}^\nu$ as the periodicity lattice of $V_{\rm per}$, we could take any other lattice as well, provided we assume $V_{\rm per}$ to be reflection invariant with respect to this lattice. However, for notational convenience, we deal only with the above easier situations. Our proofs can be easily transferred to the more complicated one. However, as we will explain, we do not know how to do one step in our proof if the continuum Schrödinger operator is replaced by a discrete one.

Let us denote by Λ_L the cube

$$\Lambda_L = \{(x_1,..., x_\nu) \mid -L/2 \leqslant x_i \leqslant L/2;\ i = 1,..., \nu\}$$

By $H_\omega^{L,D}$ (resp. $H_\omega^{L,N}$, $H_\omega^{L,\rm per}$), we denote the operator H_ω restricted to $L^2(\Lambda_L)$ with Dirichlet (resp. Neumann, resp. periodic) boundary conditions. Setting $H_1 = H_0 + V_{\rm per}$, we define $H_1^{L,D}$, etc. in a similar way. It is well known that these operators on $L^2(\Lambda_L)$ are bounded below and have a purely discrete spectrum (see, e.g., Ref. 17). We denote the eigenvalues of such an operator A counted according to their multiplicity and increasingly ordered by $\lambda_i(A), i = 1, 2,...$. We set $\rho(\lambda, A) = \#\{\lambda_i(A) \leqslant \lambda\}$. Then the integrated density of states $\rho(\lambda)$ of H_ω is defined by

$$\rho(\lambda) = \lim_{L \to \infty} \frac{1}{L^\nu} \rho(\lambda, H_\omega^{L,D})$$

It is well known that the above limit exists almost surely, is (a.s.) independent of ω, and the boundary condition chosen (see, e.g., Refs. 1, 8, and 15 and references given there).

In 1965, Lifshitz[11] argued on the basis of physical reasoning that $\rho(\lambda)$ should behave near the bottom λ_0 of the spectrum $\sigma(H_\omega)$ as $c_1 \exp[-c_2(\lambda - \lambda_0)^{-\nu/2}]$ as $\lambda \downarrow \lambda_0$. This is in sharp contrast to the periodic case ($V_\omega \equiv 0$) where $\rho(\lambda) \sim (\lambda - \lambda_0)^{\nu/2}$ (see Ref. 10). The Lifshitz behavior for random potentials was rigorously proven for certain special potentials by Benderskii and Pastur,[2] Friedberg and Luttinger,[4] Luttinger,[12] Nakao,[14] and Pastur,[16] as well as for some discretized Schrödinger operators on $l^2(\mathbb{Z}^\nu)$ by Fukushima,[5] Nagai,[13] and Romerio and Wreszinski[18] (see also Ref. 6). The works of Nakao and Pastur treat nonnegative potentials with Poisson distributed sources. This special probabilistic situation allows them to make use of the powerful

Donsker–Varadhan[3] machinery to treat a large-deviations problem. Nakao and Pastur actually proved in their case

$$\lim_{\lambda \downarrow 0} [-\lambda^{+\nu/2} \ln \rho(x)] = \gamma_\nu$$

where γ_ν is the smallest eigenvalue of $-(1/2)\Delta$ on the unit ball with Dirichlet boundary conditions. More recently, Kirsch and Martinelli[7,9] proved for rather general random potentials (for which $\lambda_0 = 0$) that

$$\lim_{\lambda \downarrow \lambda_0} \frac{\ln[\ln \rho(\lambda)^{-1}]}{\ln \lambda} = -\frac{\nu}{2}$$

For example, they treated the case of (1) with $V_{\text{per}} \equiv 0$ (for a proof see Ref. 7). In Ref. 19, Simon used their method for a lattice model using Temple's inequality (instead of Thirring's bound). We will also use this device, since Temple's inequality seems to give better results in our case. These papers exploit Dirichlet–Neumann bracketing as we will see here. The use of Neumann bracketing in this context seems to have appeared as early as the work of Harris,[20] and was also used recently by Mezincescu[21] to obtain independently results close to those in Ref. 19.

We will prove the following theorems:

Theorem 1. Let V_{per} be a periodic, reflection invariant potential and q_i i.i.d. random variables whose common distribution P_0 satisfies: $\operatorname{supp} P_0$ is compact, $\inf(\operatorname{supp} P_0) = 0$, $P_0([0, \varepsilon)) \geqslant C\varepsilon^N$, $P_0(\{0\}) \neq 1$. Let $f \in L^p(\mathbb{R}^\nu)$ satisfy $0 \leqslant f(x) \leqslant C\,|x|^{-\nu-\varepsilon}$ for large $|x|$. We set $V_\omega(x) = \sum q_i(\omega) f(x-i)$, $H_\omega = H_0 + V_{\text{per}} + V_\omega$ and $\lambda_0 = \inf \sigma(H_\omega)$. We denote by $\rho(\lambda)$ the integrated density of states for H_ω.

(i) If $f(x) \leqslant C\,|x|^{-\nu-2}$($|x|$ large), then

$$\lim_{\lambda \downarrow \lambda_0} \frac{\ln\{-\ln \rho(\lambda)\}}{\ln \lambda} = -\frac{\nu}{2}$$

(ii) If $C_1(1+|x|)^{-\alpha} \leqslant f(x) \leqslant C_2(1+|x|)^{-\alpha}$ with $\nu < \alpha < \nu + 2$, and $C_1 > 0$, then

$$\lim_{\lambda \downarrow \lambda_0} \frac{\ln\{-\ln \rho(\lambda)\}}{\ln \lambda} = -\frac{\nu}{\alpha - \nu}$$

In the one-dimensional case, we have the following:

Theorem 2. For $\nu = 1$, (i) and (ii) of Theorem 1 hold even without the assumption of reflection invariance of V_{per}.

We remark that besides the potentials V_ω as above, we can equally well treat other random potentials considered in Refs. 7 and 9. In addition, we remark that if inf supp $P_0 = a$ and $P_0([a, a+\varepsilon)) \geqslant C\varepsilon^N$, we can replace V_{per} by $V_{\text{per}}(x) + a \sum_{i \in Z^\nu} f(x-i)$ and q_i by $q_i - a$ and reduce to the case considered here. In particular, the purely random case ($V_{\text{per}} = 0$) but with inf supp $P_0 \neq 0$ should be thought of as a periodic plus positive random situation.

2. THE UPPER BOUND

Now, we turn to the proofs of the upper bounds in Theorems 1 and 2. Suppose that $\tilde{V}(x) \leqslant V(x)$. Then $\rho(\lambda) \leqslant \tilde{\rho}(\lambda)$ if $\rho(\lambda)$ and $\tilde{\rho}(\lambda)$ denote the integrated density of states for $H_0 + V$ and $H_0 + \tilde{V}$, respectively. Therefore we can suppose that supp $f \subset \Lambda_1$ for the upper bound.

For the situation (i), this replacement (of f by f times the characteristic function of Λ_1) is possible, but for case (ii) we will need to exploit the tails of f.

Following Ref. 9, we estimate

$$\rho(\lambda) \leqslant E\left[\frac{\rho(\lambda, H_\omega^{L,N})}{L^d}\right] \leqslant \frac{\rho(\lambda, H_1^{L,N})}{L^d} P(\lambda_1(H_\omega^{L,N}) \leqslant \lambda) \tag{2}$$

for any L. The first inequality comes from Neumann bracketing, and the second by noting that $\rho(\lambda, H_\omega^{L,D}) \leqslant \rho(\lambda, H_1^{L,D})$ for any ω and that $\rho(\lambda, H_\omega^{L,D}) = 0$ if $\lambda_1(H_\omega^{L,N}) > \lambda$. Hence we have to estimate $P(\lambda_1(H_\omega^{L,N}) \leqslant \lambda)$ from above, which will be given by an estimate of $\lambda_1(H_\omega^{L,N})$ from below. Following Ref. 19, we do this estimate by Temple's inequality.

We state this inequality for the reader's convenience; for a proof, we refer to Reed and Simon.[17]

Proposition 1. Let H be a self-adjoint operator which is bounded below. Suppose that $\lambda_1(H) < \lambda_2(H)$ and $\mu \leqslant \lambda_2(H)$. If ψ is a normalized vector in the domain of H such that $\langle \psi, H\psi \rangle < \mu$, then

$$\lambda_1(H) \geqslant \langle \psi, H\psi \rangle - \frac{\langle \psi, H^2\psi \rangle - \langle \psi, H\psi \rangle^2}{\mu - \langle \psi, H\psi \rangle}$$

Since $V_\omega \geqslant 0$, we have that $\lambda_2(H_1^{L,N}) \leqslant \lambda_2(H_\omega^{L,N})$ so that $\lambda_2(H_1^{L,N})$ may serve as the μ in Temple's inequality.

Before we apply Temple's inequality in the proof of Proposition 3, we need the following estimate proven in Ref. 10:

Proposition 2. Suppose that V_1 is a reflection invariant periodic potential. Then

$$\lambda_2(H_1^{L,N}) - \lambda_1(H_1^{L,N}) \geqslant \alpha L^{-2}$$

for a constant $\alpha > 0$.

The proof in Ref. 10 uses the fact that, because of the reflection symmetry, the ground state $\psi_1^{L,N}$ of $H_1^{L,N}$ is, at the same time, the ground state ψ_0 of $H_1^{L,\mathrm{per}}$. This observation also implies that $\lambda_1(H_1^{L,N}) = \lambda_1(H_1^{L,\mathrm{per}}) = \inf(\sigma(H_1))$. In particular, $\lambda_1(H_1^{L,N})$ is independent of L, so by adding a constant, we may suppose that $\lambda_1(H_1^{L,N}) = 0$ for all L. Moreover, if ψ_0 denotes the normalized positive ground state of $H_1^{1,\mathrm{per}}$ extended to $\mathbb{R}^\nu$ periodically, then $(1/L^{\nu/2})\,\psi_0(x)$ is "the" normalized ground state of $H_1^{L,N}$. For later use, we define $f_1 = \int_{\Lambda_1} f(x)\,|\psi_0(x)|^2\,dx$ and $f_2 = \int_{\Lambda_1} f(x)^2\,|\psi_0(x)|^2\,dx$. Without loss of generality, we assume that $f_1 = 1$.

We note that the arguments in Ref. 10 exploit the fact that $H_0 + V$ can be written as a Dirichlet form $\langle f\psi_0, (H_0 + V - \inf\sigma(H_1))\, f\psi_0\rangle = \langle (\nabla f)\,\psi_0, (\nabla f)\,\psi_0\rangle$, something that does not extend to the discrete case. Thus, until Proposition 2 is extended to the discrete case, we cannot extend our proof to that case.

Now, we choose $L = [(\beta\lambda)^{-1/2}]$ where $[x]$ denotes the largest integer $\leqslant x$, β will be chosen later. By Proposition 2, we have $\lambda_2(H_1^{L,N}) \geqslant \alpha\beta\lambda$. The key for our estimate of $P(\lambda_1(H_\omega^{L,N}) < \lambda)$ is the following.

Proposition 3. Suppose $\beta \geqslant \beta_0$ for a constant β_0 (independent of L). If $\lambda_1(H_\omega^{L,N}) < \lambda$, then

$$\#\{i \in \Lambda_L \mid q_i(\omega) < 4\lambda\} > \tfrac{1}{2}L^\nu$$

Proof. We want to use Proposition 1 (Temple's inequality) with $\mu = \lambda_2(H_1^{L,N}) \leqslant \lambda_2(H_\omega^{L,N})$ and $\psi(x) = (1/L^{\nu/2})\,\psi_0(x)$. To ensure $\langle\psi, H\psi\rangle < \mu$, we set $\tilde{q}_i(\omega) = \min\{q_i(\omega), 8\lambda\}$ and $\tilde{V}_\omega(x) = \sum \tilde{q}_i(\omega)\, f(x-i)$, $\tilde{H}_\omega = H_1 + \tilde{V}_\omega$. We have

$$\begin{aligned}\langle\psi, \tilde{H}_\omega^{L,N}\psi\rangle &= (1/L^\nu)\langle\psi_0, \tilde{V}\psi_0\rangle_{L^2(\Lambda_L)} \\ &= \frac{1}{L^\nu}\sum \tilde{q}_i(\omega)\int_{\Lambda_L} f(x-i)\,|\psi_0(x)|^2\,dx = \frac{1}{L^\nu}\sum_{i\in\Lambda_L}\tilde{q}_i(\omega) \leqslant 8\lambda\end{aligned}$$

Thus, taking $\beta \geqslant \beta_0$ large enough, we can ensure that $(8 + 32f_2)\,\lambda \leqslant \lambda_2(H_1^{L,N})$ by Proposition 2. Therefore we may apply Temple's inequality to $\tilde{H}_\omega^{N,L}$ and obtain

$$\begin{aligned}\lambda_1(H_\omega^{L,N}) \geqslant \lambda_1(\tilde{H}_\omega^{L,N}) &\geqslant \frac{1}{L^\nu}\sum_{i\in\Lambda_L}\tilde{q}_i(\omega) - \frac{(1/L^\nu)\sum\tilde{q}_i^2(\omega)\,f_2}{32f_2\lambda} \\ &\geqslant \frac{1}{L^\nu}\sum_{i\in\Lambda_L}\tilde{q}_i(\omega)\left(1 - \frac{8\lambda}{32\lambda}\right)\end{aligned}$$

Now, if the conclusion of the proposition was wrong, then the right-hand side of the above inequality would be larger than $\frac{3}{2}\lambda$ in contradiction to the assumption. ∎

It is a standard result of probability theory that the event $\#\{i \in \Lambda_L \mid q_i < 4\lambda\} > \frac{1}{2}L^\nu$ has exponentially small probability if λ is small enough.

Proposition 4. Suppose $\{q_i\}_{i \in \mathbb{Z}^\nu}$ are i.i.d. random variables satisfying $E(q_0) > \gamma$. Then there exists a constant $c > 0$ such that

$$P(\#\{i \in \Lambda_L \mid q_i(\omega) < \gamma\} > \tfrac{1}{2}L^\nu) \leqslant e^{-cL^\nu}$$

For a proof, see, e.g., Ref. 19.

We remark that the above proposition remains true for φ mixing, stationary $\{q_i\}$ (see, e.g., Ref. 7). Since Proposition 4 is the only probabilistic estimate we need for the upper bounds, this part of the theorems holds under the above weaker condition on $\{q_i\}$.

Combining Propositions 3 and 4, we obtain

$$P(\lambda_1(H_\omega^{L,N}) < \lambda) \leqslant e^{-cL^\nu} = e^{-c'\lambda^{-\nu/2}}$$

Since $(1/|\Lambda_L|)\, \rho(\lambda_1 H_1^{L,N}) \leqslant M$ (for λ bounded), we obtain the upper bound from inequality (2). This completes the proof of the upper bound in Theorem 1(i).

Now we turn to the upper bound in (ii). We do not cut off f to live on Λ_1. We set $|x|_m := \max_{i=1,\dots,\nu} |x_i|$. We have $f(x) \geqslant c(1+|x|_m)^{-\alpha}$ by assumption with $\alpha > \nu$, $c > 0$. We use the following crude estimate [we normalized $\lambda_1(H_\omega^{L,N}) \equiv 0$]:

$$\begin{aligned} P(\lambda_1(H_\omega^{L,N}) < \lambda) &\leqslant P(\min_{x \in \Lambda_L} V_\omega(x) < \lambda) \\ &\leqslant P\left(\sum q_i(\omega)\, c(1 + \max_{x \in \Lambda_L} |x - i|_m)^{-\alpha} < \lambda\right) \\ &\leqslant P\left(c' \sum q_i(\omega)(L + |i|_m)^{-\alpha} < \lambda\right) \\ &\leqslant P\left(tc' \sum q_i(\omega)(L + |i|_m)^{-\alpha} \lambda^{-\alpha/(\alpha-\nu)} < t\lambda^{-\nu/(\alpha-\nu)}\right) = (*) \end{aligned}$$

where t is arbitrary.

We choose $L = [\gamma\lambda^{-1/(\alpha-\nu)}]$, and set

$$I(\lambda) := \{i \mid t(\gamma\lambda^{-1/(\alpha-\nu)} + |i|_m)^{-\alpha} \lambda^{-\alpha/(\alpha-\nu)} \geqslant 1\}$$

$$(*) \leqslant P\left(\sum_{i \in I(\lambda)} c' q_i(\omega) < t\lambda^{-\nu/(\alpha-\nu)}\right)$$

$$\leqslant P\left(\sum_{i \in \Lambda_{[(1/2)t^{1/\alpha}\lambda^{-1/(\alpha-\nu)}]}} c' q_i(\omega) < t\lambda^{-\nu/(\alpha-\nu)}\right)$$

since $I(\lambda) \subset \Lambda_{[(1/2)t^{1/\alpha}\lambda^{-1/(\alpha-\nu)}]} = \Lambda$ for suitable $\gamma = \gamma(t)$.

$$(*) \leqslant e^{t\lambda^{-\nu/(\alpha-\nu)}} E(e^{-\sum_{i\in\Lambda} c' q_i(\omega)})$$

by the Tschebyscheff inequality in exponential form. Observing that $E(e^{-c'q_0}) < 1$, we have

$$(*) \leqslant \exp[t\lambda^{-\nu/(\alpha-\nu)} - |\ln E(e^{-c'q_0})| \left[\left(\frac{1}{2} t^{1/\alpha}\lambda^{-1/(\alpha-\nu)}\right)\right]^{\nu}$$

$$\leqslant \exp[-\lambda^{-\nu/(\alpha-\nu)}\left(|\ln E(e^{-c'q_0})| \frac{1}{2^\nu} t^{\nu/\alpha} - t\right)$$

for λ large enough. Choosing now t so small (positive) that

$$|\ln E(e^{-c'q_0})| \frac{t^{\nu/\alpha}}{2^\nu} - t > 0$$

we obtain the desired result.

In one dimension (without requiring reflection symmetry), we observe that the derivative of the (positive) ground state ψ_0 of $H_1^{1,\mathrm{per}}$ must vanish in Λ_1, say, at the point x_0. By periodicity, ψ_0 solves the Schrödinger equation in the interval $\tilde{\Lambda}_L = [-L/2 + x_0, L/2 + x_0]$ with Neumann boundary conditions. ψ_0 is the ground state of the Neumann problem by positivity. So we may estimate $\lambda_1(H_1^{\tilde{\Lambda}_L,N}) = \lambda_1(H_1^{\tilde{\Lambda}_L,\mathrm{per}})$ by the above methods. This proves the upper bounds in Theorem 2.

3. THE LOWER BOUND

To obtain the lower bounds, we estimate

$$\rho(\lambda) \geqslant \frac{1}{|\Lambda|} P(\lambda_1(H_\omega^{L,D}) < \lambda)$$

We set $\lambda_0 = \inf\sigma(H_1)$. Let χ be a real-valued C^∞ function with $\operatorname{supp}\chi \subset \Lambda_{3/4}$,

$\chi(x)=1$ for $x\in\Lambda_{1/2}$ and set $\chi_L(x)=\chi(x/L)$. As before, let ψ_0 be the positive ground state of $H_1^{1,\mathrm{per}}$ normalized such that $\int_{\Lambda_1}|\psi_0(x)|^2\,dx=1$.

We extend ψ_0 to a periodic function on all of $\mathbb{R}^\nu$.

Proposition 5:

$$\frac{\langle\chi_L\psi_0, H_1^{L,D}(\chi_L\psi_0)\rangle}{\langle\chi_L\psi_0,\chi_L\psi_0\rangle}\leqslant\lambda_0+cL^{-2}$$

Proof:

$$\frac{\langle\chi_L\psi_0, H_1^{L,D}(\chi_L\psi_0)\rangle}{\langle\chi_L\psi_0,\chi_L\psi_0\rangle}-\lambda_0=\frac{\langle(\nabla\chi_L)\,\psi_0,(\nabla\chi_L)\,\psi_0\rangle}{\langle\chi_L\psi_0,\chi_L\psi_0\rangle}$$

by the standard Dirichlet form calculation (integration by parts). But clearly for L large, $\langle\chi_L\psi_0,\chi_L\psi_0\rangle\geqslant 2^{-\nu}|\Lambda_L|$ and $\langle(\nabla\chi_L)\,\psi_0,(\nabla\chi_L)\,\psi_0\rangle\leqslant cL^{-2}\,|\Lambda_L|$. Combining these estimates, we obtain the desired result. ▌

Corollary. $\lambda_1(H_1^{L,D})-\lambda_0\leqslant cL^{-2}$.

Proof. Use $\chi_L\psi_0$ as a trial function in the mini-max principle. ▌

By adding a constant, we may suppose that $\lambda_0=0$. Henceforth, we take $L=[(2c\lambda^{-1})^{1/2}]+1$. By the above proposition and the corollary, we have

$$\begin{aligned}P(\lambda_1(H_\omega^{L,D})<\lambda)&\geqslant P\left(\frac{\langle\chi_L\psi_0, H_\omega^{L,D}(\chi_L\psi_0)\rangle}{\langle\chi_L\psi_0,\chi_L\psi_0\rangle}<\lambda\right)\\&\geqslant P\left(\frac{\langle\chi_L\psi_0, V_\omega\chi_L\psi_0\rangle}{\langle\chi_L\psi_0,\chi_L\psi_0\rangle}<\lambda/2\right)\\&\geqslant P\left(\frac{1}{L^\nu}\int V_\omega(x)\,dx<\lambda/2M\right)\end{aligned}$$

where $M=2^\nu\sup|\psi_0(x)|^2$. Since, by assumption, supp P_0 is compact, there is an A such that $|q_i(\omega)|\leqslant A$ almost surely. Moreover, we assume that $f(x)\leqslant c_0\,|x|^{-\nu-2}$ for $|x|>1$ (the case that the estimate on f holds only for $|x|>R_0$ can be handled in the same way).

Proposition 6. There are constants $c_1, c_2>0$ such that if $q_j<c_1\lambda$ for $|i-j|\leqslant c_2\lambda^{-1/2}$, then $\int_{\Lambda_1}V_\omega(x-i)\,dx<\lambda$.

Proof. Without loss of generality, take $i=0$. Set $J=c_2\lambda^{-1/2}$,

$$\int_{A_1} V_\omega(x)\,dx=\sum_{j\in\mathbb{Z}^\nu} q_j(\omega)\int_{A_1} f(x-j)\,dx$$

$$\leqslant \sum_{|j|\leqslant J} q_j(\omega)\int_{A_1} f(x-j)\,dx+Ac_0\int_{|x|>J}|x|^{-\nu-2}\,dx$$

Set $c_3=\|f\|_1$ and $c_4=\sup_J J^2\int_{|x|>J}|x|^{-\nu-2}\,dx(<\infty)$. Then, by the above,

$$\int_{A_1} V_\omega(x)\,dx\leqslant c_1c_3\lambda+c_2^{-2}Ac_0c_4\lambda$$

so we choose $c_1=\frac{1}{2}c_3^{-1}$ and $c_2=(2Ac_0c_4)^{1/2}$. ∎

Proposition 7. There are constants $D,\gamma>0$ such that

$$P\left(\frac{1}{L^\nu}\int_{A_L} V_\omega(x)\,dx<\frac{\lambda}{2M}\right)\geqslant P(q_0<D\lambda)^{\gamma L^\nu}$$

with L related to λ as above.

Proof:

$$P\left(\frac{1}{L^\nu}\int_{A_L} V_\omega(x)\,dx<\lambda/2M\right)$$

$$\geqslant P\left(\int_{A_1} V_\omega(x-j)\,dx<\lambda/2M \qquad \text{for all} \quad j\in A_L\right)$$

$$\geqslant P(q_j<\tilde{C}_1\lambda \qquad \text{for all} \quad j \text{ with } |j|_m\leqslant L+\tilde{C}_2\lambda^{-1/2})$$

by the previous proposition

$$\geqslant P(q_0<D\lambda)^{\gamma L^\nu}$$

for γ large enough. ∎

Collecting our various estimates, we end up with

$$\rho(\lambda)\geqslant M_1\lambda^{-\nu/2}P(q_0<C\lambda)^{\gamma L^\nu}$$

$$\geqslant M_2\lambda^{-\nu/2}\lambda^{\gamma' L^\nu}\geqslant M_2\lambda^{-\nu/2}e^{-M_3\lambda^{-\nu/2}\ln(\lambda^{-1})}$$

which finishes the proof of Theorems 1(i) and 2(i) since we did not use any reflection invariance.

The proofs of the second parts of the theorems go precisely along the same lines. The main change is in Proposition 6; we must pick $J=\lambda^{-1/\nu-\alpha}$ and so require L to be of the same order.

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